

Acceleration, Velocity and Displacement Spectra – Omega Arithmetic



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Accelerometers are robust, simple to use and readily available transducers.

Measuring velocity and displacement directly is not simple. In a laboratory test rig we could use one of the modern potentiometer or LVDT transducers to measure absolute displacement directly as static reference points are available. But on a moving vehicle this is not possible.

If we have an acceleration signal, $\ddot{x}(t)$ say, we may in principle integrate it to obtain velocity and in turn integrate the velocity to find the displacement. Anybody who has done this knows that this has to be carried out very carefully as there are several pitfalls. The result is an estimate of velocity and displacement versus time.

In many instances however we are not interested in the time behaviour but rather the behaviour in frequency. Obviously if we have an acceleration, $\ddot{x}(t)$, we may readily find its behaviour in frequency either by computing its spectral density, $G_{\ddot{x}\ddot{x}}(f)$, if it is a random signal or its Fourier transform, $\ddot{X}(f)$, if it is a transient. Note that in the course of computing the spectral density we take Fourier transforms of segments of the signal. The point to note is that when translating between time and frequency then it is the Fourier transform that is involved.

The question here is if we know the spectral density or the Fourier transform of the acceleration can we determine the corresponding velocity and displacement spectra without going through the formal integration in time? The answer is yes through what is known as 'omega arithmetic'. Why the name omega arithmetic? The answer is very simple. We are dealing with frequency, f , which we commonly measure in Hz (formerly cycles/second). However mathematically one uses the Greek omega character, ω , as the frequency in radians/seconds and where of course $\omega = 2\pi f$. As illustrated later the translation between acceleration, velocity and displacement spectra just involves simple multiplication and division operations with ω . Hence the name omega arithmetic.

Before deriving the relationships between the spectra it is useful to determine the interpretation of a spectral density. If we have an acceleration, $\ddot{x}(t)$, measured in (m/sec^2) then its spectral density,

$G_{\ddot{x}\ddot{x}}(f)$, will be in units of $\left[(\text{m}/\text{sec}^2)^2 / \text{Hz} \right]$. Note that the units are acceleration squared per Hz. The word density expresses the division by Hz. The spectral density is the distribution of the mean

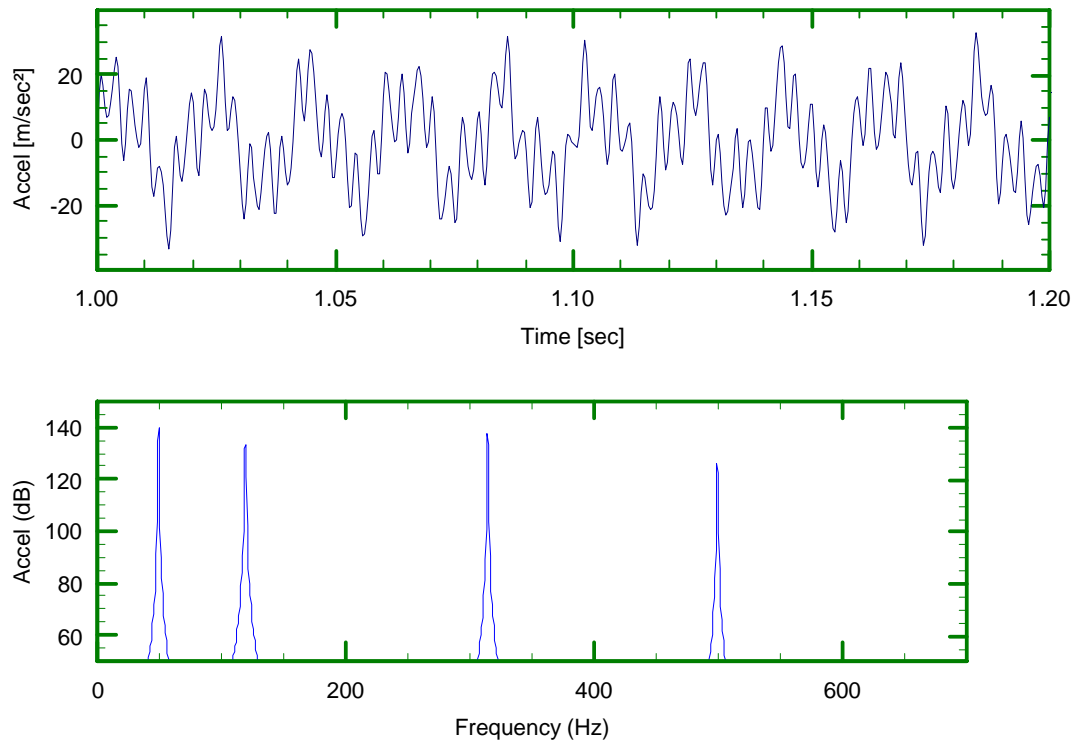
square 'energy' over frequency. If we evaluate the area between any two frequencies and take the square root then this gives us the rms level of the 'energy' in that frequency band. In DATS the RMS Spectrum over Frequency module carries out such a process. If we wished to find the rms level in third octave bands then this is just what is calculated by the Third Octave module. Conventionally third octaves are expressed as dB values, in linear form they are the rms level in the frequency band, as dB values they are known as Spectrum Levels.

Also note that if we Fourier transform the signal $\ddot{x}(t)$ to get $\ddot{X}(f)$ then the units of $\ddot{X}(f)$ are the same as those of $\ddot{x}(t)$.

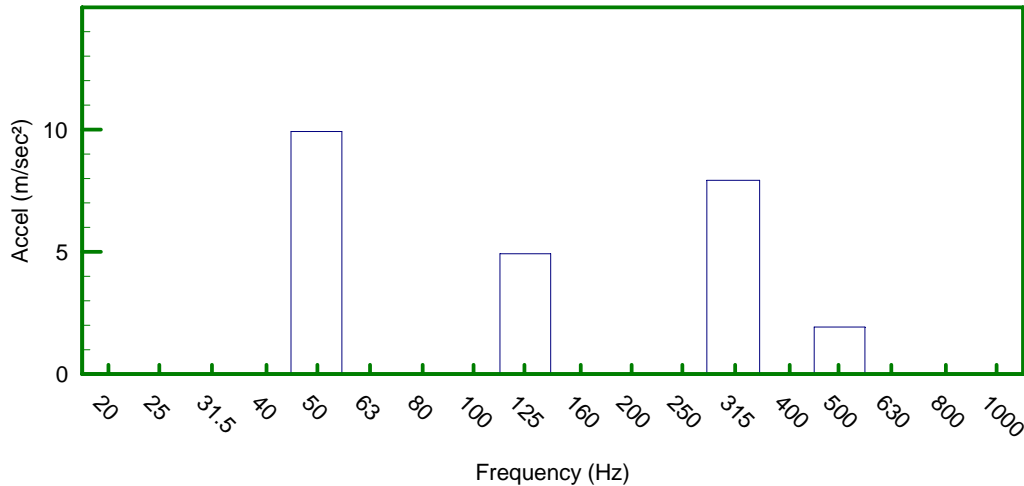
As an example consider an acceleration signal composed of four sine waves at 50, 120, 315 and 500 Hz respectively.

$$\ddot{x}(t) = 10\sqrt{2} \sin(2\pi \cdot 50 \cdot t) + 5\sqrt{2} \sin(2\pi \cdot 120 \cdot t) + 8\sqrt{2} \sin(2\pi \cdot 315 \cdot t) + 2\sqrt{2} \sin(2\pi \cdot 500 \cdot t)$$

The factor of $\sqrt{2}$ has been included explicitly as for a sine wave the relationship between the amplitude and the rms is $\sqrt{2}$. That is the rms levels of the individual sine waves are 10, 5, 8 and 2 respectively. Also for illustration purposes all the sines in the signal have zero phase. A section of the time history and the corresponding spectral density are shown below.



As expected the spectrum has four spikes centered at the relevant frequencies. If the Spectrum RMS module is used over frequency bands 40 to 60 Hz, 110 to 130 Hz, 305 to 325 Hz and 490 to 510 Hz then the calculated rms levels are, 10, 5, 8 and 2 as expected. The third octave spectrum is shown below on a linear scale rather than the usual dB scale.



Again as expected the levels in the appropriate bands are again 10, 5, 8 and 2.

Whilst we have done this for sine waves the principles apply to any signal. The essence is that the spectral density allows us to determine the rms level over a frequency range.

Returning to the main objective, the translation between acceleration, velocity and displacement spectra, we can see a possible form of relationship between acceleration and velocity merely by looking at the units. Acceleration is in (m/sec²) and velocity in (m/sec). If we note that Hz have units of (1/sec) then if we multiply a velocity by a frequency we get units of (m/sec) (1/sec) = (m/sec²). This is just illustrative. Now consider a simple sine wave velocity $\dot{x} = V_o \sin \omega t$. If we differentiate then we have $\ddot{x}(t) = -\omega V_o \cos \omega t$. Note that $\cos \omega t = \sin(\omega t + \pi/2)$. That is the acceleration is the negative of the original velocity multiplied by ω and with a 90° phase shift. If we start with the displacement $x = A \sin \omega t$ then we have $\dot{x} = \omega A \cos \omega t$ and $\ddot{x} = -\omega^2 A \sin \omega t$ and we derive $\ddot{x} = -\omega^2 x$.

To be more rigorous we need to use the Fourier representation. In a formal sense any physically realisable signal may be represented exactly by its Fourier transform. The Fourier transform is invertible. It does not either add or subtract information; it just represents it in a different way which may be easier to interpret in some circumstances. To proceed we do need to use a little mathematics. If $\ddot{x}(t)$ is the acceleration time signal and $\ddot{X}(f)$ is its infinite Fourier transform then the relationship is often written $\ddot{x}(t) \Leftrightarrow \ddot{X}(f)$ to denote that we can transform from one to the other in either direction. Using the inverse transform, frequency to time, we have by definition

$$\text{acceleration } \ddot{x}(t) = \int_{-\infty}^{\infty} \ddot{X}(f) \exp(2\pi i f t) df \quad \text{where } \exp(2\pi i f t) = e^{2\pi i f t}$$

$$\text{velocity } \dot{x}(t) = \int_{-\infty}^{\infty} \dot{X}(f) \exp(2\pi i f t) df$$

$$\text{displacement } x(t) = \int_{-\infty}^{\infty} X(f) \exp(2\pi i f t) df$$

Also we have by definition that acceleration is the rate of change of velocity

$$\ddot{x}(t) = \frac{d}{dt} [\dot{x}(t)].$$

If we substitute the relevant Fourier form of $\dot{x}(t)$ from above then we have

$$\begin{aligned}\ddot{x}(t) &= \frac{d}{dt} \left[\int_{-\infty}^{\infty} \dot{X}(f) \exp(2\pi i f t) df \right] \\ &= \int_{-\infty}^{\infty} \dot{X}(f) \frac{d}{dt} [\exp(2\pi i f t)] df \\ &= \int_{-\infty}^{\infty} 2\pi i f \dot{X}(f) \exp(2\pi i f t) df\end{aligned}$$

Comparing this with the Fourier representation of $\ddot{x}(t)$ we immediately see that

$$\ddot{X}(f) = 2\pi i f \dot{X}(f)$$

giving

$$\dot{X}(f) = \ddot{X}(f) / (2\pi i f) = \ddot{X}(f) / i\omega$$

If we know either $\dot{X}(f)$ or $\ddot{X}(f)$ then simply by multiplying or dividing by $2\pi i f$ as appropriate we may find the other type exactly.

There is one reservation and that concerns low frequencies. As f becomes zero then $\dot{X}(f)$ derived from $\ddot{X}(f)$ becomes indeterminate. So in reality there is a low frequency limit, typically below 10Hz. This is often referred to as “1/f noise”. In most dynamics situations we are interested in much higher frequencies but if it is say whole body dynamics then direct integration is needed. Fortunately it is at low frequencies where direct integration is least error prone. When we are dealing with digital systems, “low frequency” is low relative to the sample rate, say sample rate/1000.

Clearly the situation also extends to displacement so we have

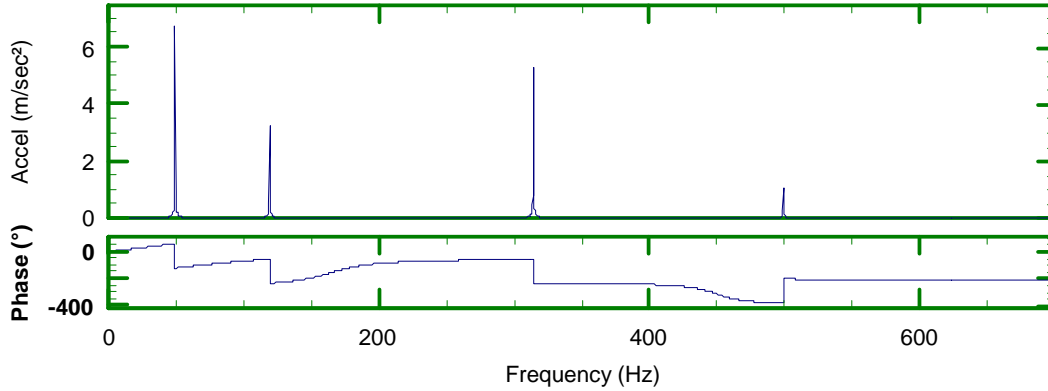
$$\ddot{X}(f) = -(2\pi f)^2 X(f) = -\omega^2 X(f)$$

This is precisely the same relationship as determined for our sine wave example.

The signs tell us that the amplitude and acceleration are 180° out of phase with each other and that the velocity is +/- 90° from the acceleration and the amplitude. Starting with an acceleration then the velocity lags by 90° and the amplitude lags by a further 90°.

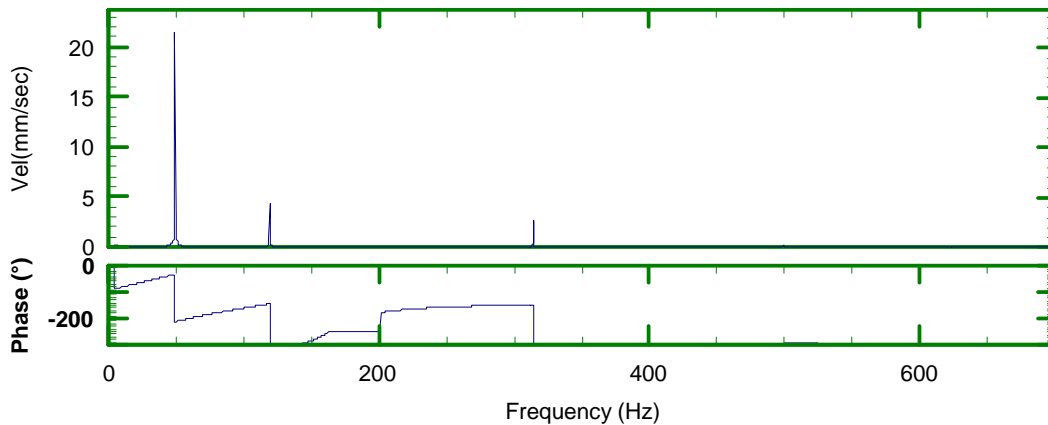
Fourier Spectra

Now let us look at the acceleration, velocity and displacement Fourier transforms where the velocity and displacement transforms were calculated using omega arithmetic from 5Hz upward. Note that all Fourier transforms shown here give half amplitudes.

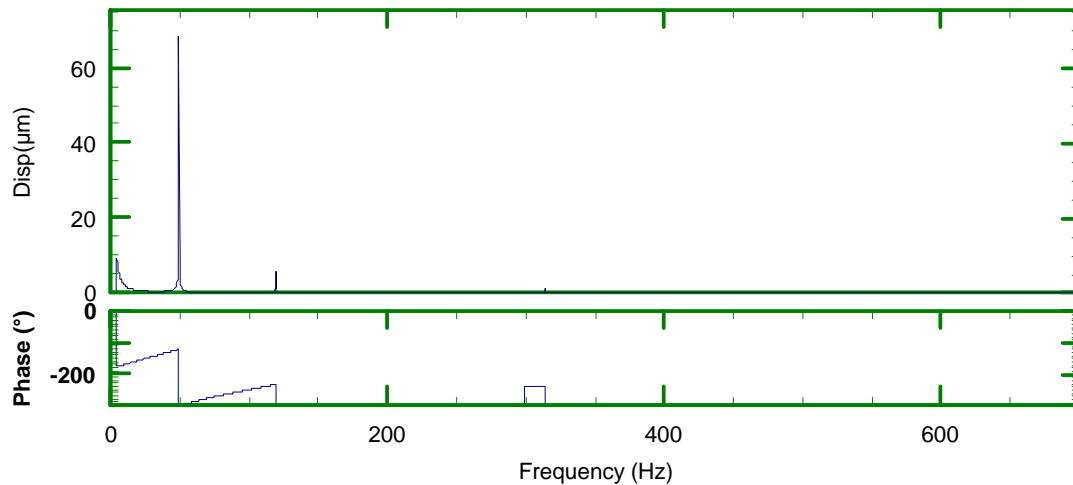


With the acceleration transform above all four frequencies are clearly present. Note that the phase starts at zero degrees. As earlier the calculated rms levels are, 10, 5, 8 and 2 as expected. If one reads “a half amplitudes directly then you will find that they all read a little low as none of them exactly match the frequencies at which the finite Fourier transform was evaluated. We discuss this artefact in another note.

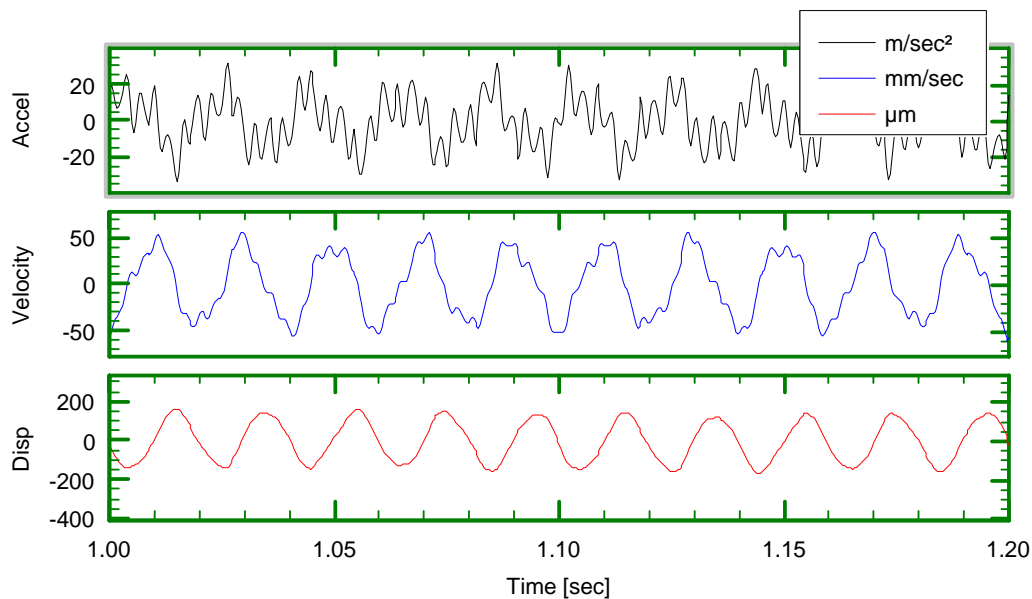
In the velocity transform below, shown in *mm/sec* not *m/sec*, a low frequency limit of 5Hz was used, which has been set deliberately too low for demonstration purposes. The amplitude of the 500Hz component is only about 0.4 *mm/sec* so it is effectively zero compared to the others! Also note that after the 5Hz “cut off” the phase starts in the -90° region.



With the displacement spectrum below we are close to having only one effective component. The phase now starts in the -180° region. Also note the obvious “1/f” noise at the low frequency end.



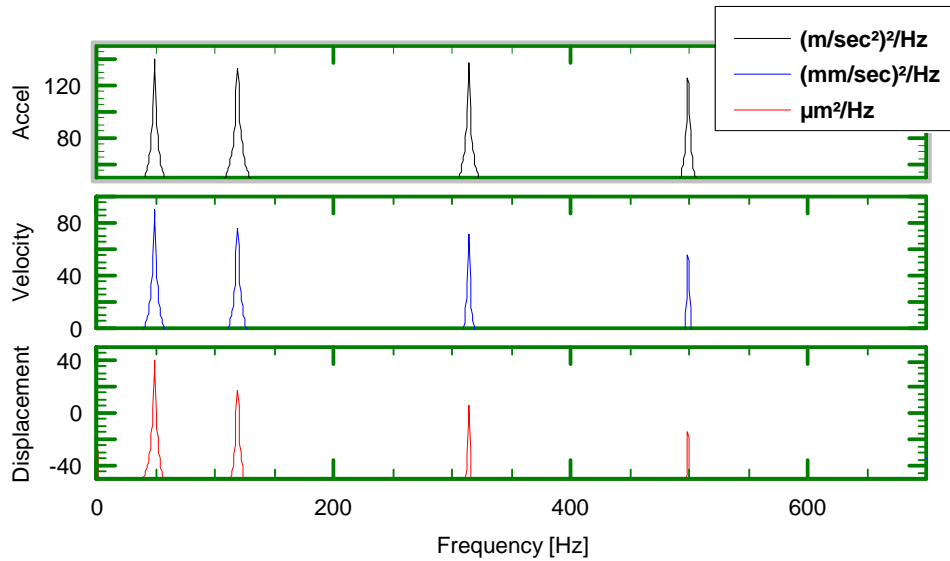
Because Fourier transforms may be Inverse Transformed then we may obtain the corresponding time signals. A section of the resulting time histories are shown below. If the entire velocity and displacement time signals were shown they would show some distortions at the beginning and end that are largely due to the “1/f” noise. .



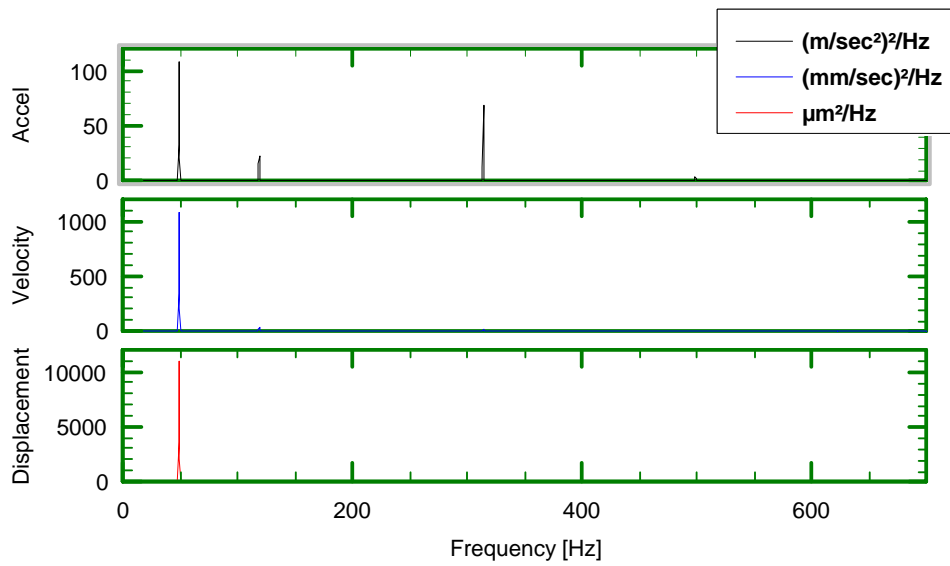
Spectral Densities

Now let us look at the acceleration, velocity and displacement spectral densities calculated using omega arithmetic. With spectral densities we are dealing with the product of two Fourier transforms so the relationships are further squared. The relationships are shown in summary below.

In the first graph below the spectral densities are shown on a dB scale. Notice how the relative height of each 'spike' changes.



If we now look on a linear scale one can only just see all four spectral lines in the acceleration spectra; the velocity and displacement spectra have effectively just one spectral line.



Fourier Transform Omega Arithmetic

Input	Required Output		
	$X(f)$	$\dot{X}(f)$	$\ddot{X}(f)$
$X(f)$	1	$1/i\omega$	$-1/\omega^2$
$\dot{X}(f)$	$i\omega$	1	$1/i\omega$
$\ddot{X}(f)$	$-\omega^2$	$i\omega$	1

Spectral Density Omega Arithmetic

Input	Required Output		
	$G_{xx}(f)$	$G_{\dot{x}\dot{x}}(f)$	$G_{\ddot{x}\ddot{x}}(f)$
$G_{xx}(f)$	1	$-1/\omega^2$	$1/\omega^4$
$G_{\dot{x}\dot{x}}(f)$	$-\omega^2$	1	$-1/\omega^2$
$G_{\ddot{x}\ddot{x}}(f)$	ω^4	$-\omega^2$	1

The standard units of acceleration, velocity and displacement are metres squared per second, m/sec^2 , millimetres per second, mm/sec and micrometres, μm . The reason for choosing these units is that most vibration occurs in the 50 Hz to 500 Hz region. By choosing the above units then the absolute values of acceleration, velocity and displacement are similar. They have exactly the same value when $2\pi f = 1000$, that is at a frequency of about 159.15Hz.

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